*Remark 2.* A remarkable property of the above estimate for the growth of the perturbations is that it was derived independently of the specific form of the second variation of the potential energy $\Pi^{(2)}$. The only prerequisite for the validity of (2.9) is the existence of a perturbation with negative $\Pi^{(2)}$ (2.2) and the truth of (2.3).

*Remark 3.* An interesting problem in determining the largest value $\Lambda^+$ of the upper bound $\Lambda$ (2.11) for all kinematically admissible fields $\xi^*(\mathbf{x})$ (2.2). Solution of this problem would make it possible to determine not only the largest values of $\lambda$ but also to ascertain the actual form of those initial data (2.10) most "dangerous" in this respect. The variational problem arising here reduces to minimizing the functional $\Pi^{(2)}$ conditional on $M = 1$.

*Remark 4.* Proofs of the instability of the above system in various special cases, using methods of spectral theory, may be found in /3, 4/. The formulation of the problem studied in /3/ is the same as that considered here, but there the surface tension forces were not taken into account. On the other hand, surface tension was considered in /4/ but only in the case of a stationary vessel. In /3, 4/ the existence of eigenvalues that yield exponential growth of the perturbations is proved, but without supplying estimates of the growth rate.

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*PMM U.S.S.R.*, Vol.53, No.4, pp.477-486, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00


# A TURBULENT VORTICAL DYNAMO*

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The possibility of the spontaneous appearance of rotational motion in a half-space above a plane, caused by bifurcation at some Reynolds number determining the intensity of the given sources of motion not giving rise to external force moments, is studied in the class of selfsimilar conical flows of an incompressible fluid of variable viscosity. The impossibility of spontaneous rotation is shown for the cases of constant viscosity and state of rest, and of weak sources of the basic flow. Examples of the bifurcations of the autorotation are constructed for an ascending, one-cell motion under the condition that there is no rotational friction, and for a two-cell motion with conditions of regularity on the axis and adhesion at the fixed plane. In these cases the motion is made up of an outer laminar flow, and a turbulent, high viscosity kernel near the axis. The examples quoted obviously model rotating astrophysical jets, the initiation of a whirlpool, and the onset of a firestorm above a plane under the action of a quadrupole heat source.

***1. The concept of a vortical dynamo.*** We shall use the name "vortical dynamo" to describe

**Prikl.Matem.Mekhan.*, 53,4,613-624,1989

the spontaneous excitation of the rotational motion of a fluid without visible sources of rotation, i.e. under the conditions when irrotational motion is known to be possible. The appearance of a vortical dynamo is connected with the fact of a straightforward bifurcation of the initial mode at some Reynolds number $\mathrm{Re} = \mathrm{Re}_*$, when the equation for the rotational component begins to admit of non-trivial solutions corresponding to a stable mode, and the mode without rotation becomes unstable. This means that in almost all cases of initial perturbations of the original mode, it will evolve with time, at fixed $\mathrm{Re} > \mathrm{Re}_*$, to a stable mode with rotation. There are two such modes in the present case, and they differ from each other in the direction of rotation only. It is precisely this phenomenon that we call here the "autorotation" or the "vortical dynamo".

Thus the vortical dynamo has three characteristic features: 1) there is an excitation threshold (relative to Re), 2) the direction of rotation is arbitrary and depends on the sign of the initial rotational perturbation, and 3) when the motion has been established, the initial perturbation is completely forgotten and the intensity of rotation does not depend on it. From the point of view of the above properties, the problem of the vortical dynamo is completely analogous to the well-known problem of the MHD dynamo in which the magnetic field is spontaneously generated during the motion of a conducting fluid without external field sources. It is this that suggested the name vortical dynamo.

The problem of whether a steady-state vortical dynamo is possible under the conditions of the axisymmetric problem, is not trivial. In the related MHD problem Cowling's theorem /1/ is encountered, according to which spontaneous generation of a magnetic field is not possible in the axisymmetric case. An analogous theorem for the circulation can be proved for an axisymmetric flow of a homogeneous liquid under the condition of adhesion. It is clear that conditions of adhesion, and in particular the fact that the rotational component of velocity if equal to zero, form an obstacle to the rotation of the liquid. If, in spite of this, an autorotation appears, then we shall call it "strong". In contrast, a spontaneously occurring rotation when there is no friction at the boundary, will be called "weak". We cannot have a strong rotation in a uniform liquid. If on the other hand the conditions of zero rotational friction are specified on a part of the boundary, then autorotation is possible, as shown in a recently discovered example*. (*Gol'dshtik M.A. and Yavorskii N.I., Flow between a rotating porous disc and a plane. Preprint 152, Novosibirsk, ITF, Siberian Branch, Academy of Sciences of the USSR, 1987). The appearance of autorotation connected with the previous loss of symmetry of the initial mode caused by hydrodynamic instability, was studied in /2/.

In the present paper we disclose a bifurcation of autorotation in the axisymmetric flow of an inhomogeneous liquid of variable viscosity. If we regard the variation in viscosity as the result of turbulence, then using the Boussinesq model we can construct examples of the excitation of weak and strong autorotation for a number of flows with a turbulent kernel, or in other words, to present a model of a turbulent vortical dynamo.

We will carry our investigation in the class of conically symmetric flows, in which the velocity vector decreases in modulo in inverse proportion to the spherical radius in the direction away from the origin of coordinates. This class includes the well-known solutions for the flow in a diffuser, the solution for an impulsive source of an axisymmetric submerged jet /3/, and the jets emerging from the tip of a cone /4/, the solutions for swirling jets /5, 6/, and a number of others /7-9/*. (*See also: Gol'dshtik M.A. and Shtern V.N., Induced jets and critical phenomena in viscous flows. Preprint 159, Novosibirsk, ITF Siberian Branch, Academy of Sciences of the USSR, 1987; Gol'dshtik M.A. and Shtern V.N., Selfsimilar problems of thermal convection. Preprint 170, Novosibirsk, ITF, Siberian Branch, Academy of Sciences of the USSR, 1988.).

All flows in question have a distinct feature, namely their selfsimilarity related to the absence of a characteristic scale of length, and to specifying, as the sources of motion, the quantities with dimensions of kinematic viscosity $\nu$ or of its degree. In the simplest case only one such quantity $Q$ is specified, determining the Reynolds number $\mathrm{Re} = Q/\nu$. The following assertion holds for the motions of the type discussed here /10/: if a motion exists, then it must be selfsimilar.

*2. Basic equations.* We shall consider a turbulent flow stationary in the mean. We shall use the Boussinesq model to close the Reynolds equations, introducing the effective turbulent kinematic viscosity $\nu_t$, and we define the total relative viscosity by the relation $\varepsilon = (\nu + \nu_t)/\nu$. Then the equations for the mean velocity field $\mathbf{u}$ can be written in the form /9/

$$(\mathbf{u}, \nabla)\,\mathbf{u} = -\rho^{-1}\nabla p + \nu\varepsilon\Delta\mathbf{u} + \nu\nabla\varepsilon\cdot D + \mathbf{F}, \quad \nabla\mathbf{u} = 0 \tag{2.1}$$

($D$ is the doubled deformation rate tensor and $\mathbf{F}$ is the vector of external forces). We shall assume that the turbulent motion is selfsimilar in the mean and belongs to the class with conical symmetry, i.e.

$$u_R = -\frac{\nu}{R}\,y'(x), \quad u_\theta = -\frac{\nu}{R}\,\frac{y(x)}{(1-x^2)^{1/2}}, \quad u_\varphi = \frac{\nu}{R}\,\frac{\Gamma(x)}{(1-x^2)^{1/2}} \tag{2.2}$$

$$\frac{p}{\rho}=\frac{\nu^2}{R^2}q(x)+\text{const},\quad x=\cos\theta$$

where $R,\theta,\varphi$ are the spherical coordinates, $\theta$ is the angle between the positive $z$ semi-axis and the radius vector, and $\varphi$ is the azimuthal angle on which the flow does not depend. Here and henceforth a prime denotes a derivative with respect to $x$. Conical symmetry is possible only in the case when the external force has the following representation:

$$\mathbf{F}=\rho\nu^2R^{-3}\mathbf{f}(x) \tag{2.3}$$

and the turbulent viscosity depends only on the angle $\theta$, $\varepsilon=\varepsilon(x)$ /8/.

We will further assume that $\mathbf{f}=\{f_R, f_\theta, 0\}$, which means that the external forces do not produce rotational momentum. The symmetric tensor $D$ has the following components:

$$D_{RR}=\frac{2\nu}{R^2}y',\quad D_{R\theta}=\frac{\nu(1-x^2)^{1/2}}{R^2}\left(y''+\frac{2y}{1-x^2}\right),\quad D_{R\varphi}=-\frac{2\nu}{R^2}\frac{\Gamma}{(1-x^2)^{1/2}} \tag{2.4}$$

$$D_{\theta\theta}=\frac{2\nu}{R^2}\frac{xy}{1-x^2},\quad D_{\theta\varphi}=-\frac{\nu}{R^2}\left(\Gamma'+\frac{2x\Gamma}{1-x^2}\right),$$

$$D_{\varphi\varphi}=-\frac{2\nu}{R^2}\left(y'+\frac{xy}{1-x^2}\right)$$

The vector $\nabla\varepsilon$ has a unique non-zero component

$$(\nabla\varepsilon)_\theta=-(1-x^2)^{1/2}\varepsilon'R^{-1} \tag{2.5}$$

Substituting relations (2.2)-(2.5) into Eq.(2.1), we obtain the following equation for $\Gamma$:

$$\varepsilon(1-x^2)\Gamma''-y\Gamma'+\varepsilon'[(1-x^2)\Gamma'+2x\Gamma]=0 \tag{2.6}$$

The same substitution yields, after eliminating the dimensionless pressure $q$, the relation

$$(\varepsilon z')''={}^1/_2(y^2)'''+2(1-x^2)^{-1}(\varepsilon'z+\Gamma\Gamma')+f''' \tag{2.7}$$

where the following notation is introduced:

$$z=(1-x^2)y'+2xy,\quad f'''=f_R'-2f_\theta(1-x^2)^{-1/2} \tag{2.8}$$

The function $f$ is determined, apart from three integration constants. It is introduced in accordance with the method used in /7/, so that Eq.(2.7) is integrated three times. We adopt arbitrarily the conditions

$$f(1)=f'(1)=f''(1)=0 \tag{2.9}$$

Note that the Eqs.(2.6) and (2.7) in /9/ are incorrectly written.

Multiplying both sides of Eq.(2.7) by $1-x^2$ and integrating, we obtain

$$\varepsilon(1-x^2)^2y'''+\varepsilon'(1-x^2)[(1-x^2)y''+2y]= \tag{2.10}$$

$${}^1/_2(1-x^2)(y^2)''+x(y^2)'-y^2+\Gamma^2+(1-x^2)f''+2xf'-2f$$

The arbitrary constant is put equal to zero, since it is assumed that the axis $x=1$ lies inside the domain of the flow. This means that the solution is analytic at $x=1$ and, especially, that the conditions $y(1)=0$ and $\Gamma(1)=0$ hold, the latter following directly from (2.2).

3. ***Formulation of the boundary value problems.*** Various boundary conditions can be formulated for Eqs.(2.6) and (2.10), specifying, for example, a velocity field of the vortex source-type in the plane $x=0$, and the conditions of analyticity on the axis $x=1$. In this formulation the inhomogeneous boundary conditions will serve, in addition to the mass force $\mathbf{f}$, as the source of the motion. However, a single case exists in which the problem is formulated in an unbounded region, when there are no mass forces, with the conditions of regularity on the semi-axes $x=\pm1$, and the motion is induced by a source of momentum at the origin of coordinates. This is the Landau jet /3/ for which the solution can be written in analytical form when $\Gamma\equiv0$ and $\varepsilon\equiv1$, in the form $y=2(1-x^2)/(A_0-x)$ where $A_0$ is an arbitrary constant which can be related to the momentum of the jet. Thus, the case of a Landau jet is spectral when there is a non-trivial solution with homogeneous conditions at $x=\pm1$. If we formulate an analogous homogeneous problem in the half-space $x\geqslant0$, then, provided that the condition of adhesion on the plane holds, there will be no non-trivial solutions.

We have two problems here. Is it not possible to find a distribution of the viscosity $\varepsilon(x)$, such that when there are no mass forces, firstly, a selfsimilar solution of the problem of a jet can be found, and secondly, a non-trivial solution of Eq.(2.6) exists, regular at $x=1$ and satisfying one of the homogeneous conditions, namely $\Gamma'(0)=0$ (weak autorotation), or $\Gamma(0)=0$ (strong autorotation)? We note that when the second problem is solved satisfactorily, the problem of the non-trivial solvability of Eq.(2.10) vanishes since the equation becomes inhomogeneous when $\Gamma \not\equiv 0$.

It transpires, that a selfsimilar ascending jet cannot satisfy the conditions of adhesion for any form of the relation $\varepsilon(x)$, and the bifurcation of autorotation is thereby eliminated. In order to change this situation we must introduce into the problem other sources of motion which are not point sources.

*4. Non-existence of solutions under the conditions of adhesion.* We shall first turn out attention to Eq.(2.6). If $\varepsilon \equiv 1$, then integration shows at once that the function $\Gamma(x)$ is monotonic and a non-trivial solution is impossible. We convert Eq.(2.6) identically to the form

$$[\varepsilon(1-x^2)\Gamma']' + 2(x\varepsilon\Gamma)' = 2\varepsilon\Gamma + y\Gamma' \tag{4.1}$$

For weak or strong autorotation it is necessary that the function $\Gamma(x)$ has one or several extremal points at which $\Gamma'=0$. Let us consider the point $x_0$ nearest to the right end. We can assume without loss of generality, that $\Gamma(x_0)>0$. It is clear that under these conditions the function $\Gamma(x)$ decreases monotonically over the interval $x_0 \leqslant x \leqslant 1$ while remaining positive, and $\Gamma(1)=0$. Integrating the relation (4.1) from $x_0$ to $x$ and using the condition that $\Gamma(x)$ is analytic when $x=1$, we obtain

$$\varepsilon(1)\Gamma(1) = x_0\varepsilon(x_0)\Gamma(x_0) + \int_{x_0}^{1}(\varepsilon\Gamma + {}^1/_2 y\Gamma')\,dx \tag{4.2}$$

Let $\varepsilon(x)>0$. Then from (4.2) it follows that the condition $\Gamma(1)=0$ cannot be satisfied if an ascending flow with $y(x)\leqslant 0$ exists near the axis in the interval $(x_0, 1)$, since we have $\Gamma'(x)\leqslant 0$ in that interval. It is clear that the above argument remains valid for sufficiently small $y(x)\geqslant 0$, and, in particular, for motions at small Reynolds numbers when the last term in (4.1), representing the inertial forces, is neglected.

To obtain further results, we shall write Eq.(2.6) in the selfconjugate form

$$(\varepsilon E\Gamma')' + 2\frac{\varepsilon' E x}{1-x^2}\Gamma = 0;\quad E = \exp\left[-\int_0^x \frac{y\,dx}{\varepsilon(1-x^2)}\right] \tag{4.3}$$

From (4.3) it follows at once, that when $\varepsilon'<0$, neither of the conditions $\Gamma(0)=0, \Gamma'(0)=0$ can be satisfied, i.e. selfconjugation is impossible. To confirm this, it is sufficient to integrate (4.3) from 0 to $x$ and obtain the inequality $\Gamma'(x)>0$.

Relation (4.3) will be identically transformed to the form

$$\{\varepsilon E[(1-x^2)\Gamma' + 2x\Gamma]\}' = 2\varepsilon\Gamma(xE)'$$

Integrating from 0 to $x$ yields

$$\Gamma'(0) = -2\int_0^1 \varepsilon\Gamma(xE)'\,dx$$

From this it follows that when $(xE)'>0$, autorotation is impossible. The last inequality yields the sufficient condition for there to be no autorotation

$$xy \leqslant \varepsilon(1-x^2) \tag{4.4}$$

Thus we can hope to obtain autorotation only under the condition that an ascending motion of a high viscosity liquid takes place near the axis.

In what follows, we shall assume that $y'(0)=\mathrm{Re}\geqslant 0, y(0)=0,\ \varepsilon(x)>0, \varepsilon'(x)\geqslant 0,\ \varepsilon(0)=1,\ \varepsilon'(0)=0.$ The last condition is plausible, and is used here only in order to simplify the proof.

Let us introduce the functions $H$ and $G$ by means of the relations

$$H = G + f,\ G''' = 2\Gamma\Gamma'/(1-x^2),\ G(1) = G'(1) = G''(1) = 0 \tag{4.5}$$

and integrate the relation (2.7) three times. This yields

$$\varepsilon Z = {}^1/_4 y^2 + \int_0^x \left[\frac{(x-t)^2}{1-t^2} + 1\right]\varepsilon' Z\,dt + H + {}^1/_2 C_1 x^2 + C_2 x + C_3 \tag{4.6}$$

To find the constants $C_1, C_2, C_3$ appearing after every integration stage, we put $x=0$ and obtain

$$C_1 = 2\mathrm{Re} - 2f(0) + \Gamma^2(0) - G''(0) \tag{4.7}$$

$$C_2 = Z'(0) - H'(0) = y''(0) - H'(0) \tag{4.8}$$

$$C_3 = \mathrm{Re} - H(0) \tag{4.9}$$

We shall show that without rotation $(\Gamma \equiv 0)$, and when there is no field of mass forces $(f \equiv 0)$ selfsimilar motion with condition of adhesion on the plane $x = 0$ is impossible.

Having satisfied in (4.6) the condition $Z(1) = 0$, we obtain

$$C_2 = -2\,\mathrm{Re} - 2\int_0^1 \frac{\varepsilon' z}{1+t}\,dt \tag{4.10}$$

Substituting relations (4.7), (4.9) and (4.10) into (4.6), we obtain

$$\varepsilon Z = {}^1/_2 y^2 + (1-x)^2 \int_0^x \frac{\varepsilon' Z}{1-t^2}\,dt - 2x \int_x^1 \frac{\varepsilon' Z}{1+t}\,dt + \mathrm{Re}\,(1-x)^2 \tag{4.11}$$

We shall show that when $\mathrm{Re} \geqslant 0$, the function $Z(x)$ has no zeros in the interval $(0, 1)$. Let us assume the opposite, that $Z(x)$ has one root $Z(x_0) = 0$. Then, putting $x = x_0$, in (4.11) we obviously arrive at a contradiction since all terms on the right-hand side are positive. Let now $Z(x)$ have two zeros: $Z(x_1) = Z(x_2) = 0$. Substituting $x = x_1$ and $x = x_2$ into (4.11) and taking the difference of these two relations, we can easily arrive at a contradiction. In the same manner we can prove that $Z(x)$ cannot have an arbitrary number of zeros. When the analyticity is assumed, this means that $Z(x) \geqslant 0$.

From the definition (2.8) of the function $Z(x)$ we can obtain

$$(1-x^2)^2[y/(1-x^2)]' = Z \geqslant 0$$

therefore $y(x) \geqslant 0$. Thus, when there are no mass forces the converging motion of the material of the plane causes, when $u_R(0) < 0$ or according to (2.2) $y'(0) = \mathrm{Re} > 0$, only ascending motion of the fluid with $y(x) > 0$. This implies, in particular, that the conditions of adhesion $y(0) = y'(0) = \mathrm{Re} = 0$ cannot be satisfied. Indeed, in this case we should have $y''(0) > 0$ and the relation $y''(0) = 0$ is inadmissible since in this case we have, according to (2.10), $y \equiv 0$. This means that according to (4.8), $C_2 > 0$ and this contradicts relation (4.10).

We note that the above arguments also imply the impossibility of the existence of a double-sided jet with condition of symmetry $y''(0) = 0$. Indeed, for such a jet we have, according to (4.8), $C_2 = 0$, while according to (4.7) and (4.9) $C_1 = 2C_3 > 0$. But in this case it follows from (4.6) that condition $Z(1) = 0$ cannot be satisfied.

**5. *Weak autorotation in a model of a selfsimilar turbulent jet.*** Let the motion of a viscous fluid be generated by a source of abundance $Q$ in the plane, so that $u_R = -Q/R$ when $x = 0$. Then $y(0) = 0$, $y'(0) = \mathrm{Re}$. If the fluid is homogeneous, the flow can be described by the analytic solution due to Squire /4/, which is characterized by the presence of an induced jet near the axis, of intensity which becomes infinite when the value of Re reaches its critical value $\mathrm{Re}_* = 7.67$ /11/. Let us consider an inhomogeneous fluid with piecewise-constant viscosity

$$\varepsilon(x) = \begin{cases} \varepsilon_1, & 0 \leqslant x < x_k \quad \text{(zone 1)} \\ \varepsilon_2, & x_k < x \leqslant 1 \quad \text{(zone 2)} \end{cases}$$

Assuming that the flow in zone 1 is laminar, we put $\varepsilon_1 = 1$, and in zone 2 near the axis we write $\varepsilon_2 = \mathrm{const} = \beta = 1 + \nu_t/\nu \geqslant 1$. In this case we must assume in all previous relations that $\varepsilon' = 0$ everywhere except at the point $x = x_k$. At this point we must set the conditions of continuity of the velocities $u_\xi$ and of the components of the momentum flux tensor $\Pi_{\theta\xi}$, where

$$\Pi_{\xi\eta} = \rho u_\xi u_\eta + p\delta_{\xi\eta} - \rho\nu D_{\xi\eta}; \quad \xi, \eta = R, \theta, \varphi \tag{5.1}$$

It is convenient to itroduce for the zones 1 and 2 their own variables, by putting

$$\begin{gathered} y = y_1, \quad \Gamma = \Gamma_1, \quad H = H_1, \quad 0 \leqslant x < x_k \\ y = \beta y_2, \quad \Gamma = \beta\Gamma_2, \quad H = \beta H_2, \quad x_k < x \leqslant 1 \end{gathered} \tag{5.2}$$

Then, taking into account (2.8), we can write Eqs. (4.6) and (2.6) in the form

$$(1-x^2)\,y_i' + 2xy_i = {}^1/_2 y_i^2 + Ax - A_1x^2 + A_2 + H_i \tag{5.3}$$

$$(1-x^2)\,\Gamma_i'' = y_i\Gamma'_i; \quad i=1,2$$

Using relations (2.2), (2.4) and (5.3) we obtain, in accordance with (5.1),

$$\Pi_{R\theta} = -\Pi\,(1-x^2)^{-1/2}(H_i' + A - 2A_1x), \tag{5.4}$$
$$\Pi_{\theta\theta} = -\Pi\,(1-x^2)^{-1}[H_i - {}^1/_2\,(1-x^2)\,H_i'' + {}^1/_2\Gamma_i^2 + Ax + A_1\,(1-2x^2) + A_2]$$
$$\Pi_{\varphi\theta} = \Pi\,(1-x^2)^{-1}[(1-x^2)\,\Gamma_i' + 2x\Gamma_i - y_i\Gamma_i]; \quad \Pi = \rho\nu^2/R^2$$

The requirements of regularity in zone 2 at $x=1$ leads to the relations $A = 2A_1$, $A_2 = -A_1$. Therefore we have, in accordance with (5.3),

$$(1-x^2)\,y_2' + 2xy_2 = {}^1/_2y_2^2 - A_1\,(1-x)^2 + H_2 \tag{5.5}$$

After differentiating Eq.(5.5) we see that when $y_2\,(1)=0$, the quantity $y_2'\,(1)$ is basically indeterminate. It should be specified, together with $A_1$ and $\Gamma_2'\,(1)$, as an arbitrary parameter necessary for integrating Eq.(5.5) from $x=1$ to $x=x_k$.

For the zone 1 we write

$$(1-x^2)\,y_1' + 2xy_1 = {}^1/_2y_1^2 + Bx - B_1x^2 + B_2 + H_1 \tag{5.6}$$

We shall use the conditions of compatibility at the point $x = x_k$ to find $B$, $B_1$, $B_2$. Taking into account (5.2), we have

$$y_1 = \beta y_2, \quad y_1' = \beta y_2', \quad \Gamma_1 = \beta\Gamma_2 \tag{5.7}$$

Using relations (5.4) and taking into account (5.7), we obtain

$$B - 2x_kB_1 = \beta^2\,[2A_1\,(1-x_k) + H_2'] - H_1' \tag{5.8}$$

$$Bx_k + B_1\,(1-2x_k^2) + B_2 = \beta^2\,[2A_1x_k\,(1-x_k) + H_2 - {}^1/_2\,(1-x_k^2)\,H_2''] - H_1 + {}^1/_2\,(1-x_k^2)\,H_1'' \tag{5.9}$$
$$(1-x_k^2)\,\Gamma_1' + 2x_k\Gamma_1 = \beta^2\,[(1-x_k^2)\,\Gamma_2' + 2x_k\Gamma_2] \tag{5.10}$$

Moreover, according to (5.6) we have

$$Bx_k - B_1x_k^2 + B_2 = (1-x_k^2)\,y_1' + 2x_ky_1 - {}^1/_2y_1^2 - H_1 \tag{5.11}$$

The parameters $B$, $B_1$, $B_2$ are found from the system of Eqs.(5.8), (5.9) and (5.11), taking (5.7) into account, after solving Eq.(5.5). This enables us to integrate Eq.(5.6) from $x_k$ to $x=0$, using the first condition of (5.7). Let the parameter $A_1$ be determined by the requirement that $y\,(0)=0$. Then, using the given $y_2'\,(1)$, $\Gamma_2'\,(1)$, $\beta$, $x_k$ we obtain the complete solution including the quantity $\mathrm{Re} = y_1'\,(0)$.

The problem of specifying the parameters $y_2'\,(1)$ and $x_k$ demands that we turn to experimental data and physical models. We know that the turbulent flow at the kernel of the jet is selfsimilar and practically independent of the method of generating it. According to the results of /12/ the longitudinal velocity on the axis of the turbulent jet is given by the expression $u_R = 3K/(8\pi\nu_tR)$, where $K = (\nu_t/0.0161)^2$ is the momentum of the jet. The above relations yield

$$Ru_R/\nu_t = -y_2'\,(1) = 460.5 \tag{5.12}$$

We can eliminate the parameter $x_k$, provided that we assume that the boundary $x_k$ coincides with the point of the maximum of the function $y\,(x)$ characterizing the ejection capacity of the jet. We have an increasing ejection of the surrounding fluid in the outer region $x < x_k$, while within the cone $x > x_k$ the flow rotates in the direction of the axis, and this is the most likely reason for its intense turbulence.

We shall therefore use $y'\,(x_k) = 0$ as the tentative hypothesis. As regards the parameter $\beta$, the relation $\beta\,(\mathrm{Re})$ will be determined by (5.12) and the parameter can vary over a wide range, beginning with unity at the instant of turbulence begins.

According to the model used, the jet will remain laminar as Reynolds number $\mathrm{Re} = Q/\nu$ increases, until the value $y_2'\,(1) = -460.5$ is reached, with the corresponding value of $\mathrm{Re}^* = 7.56$. After this an increase in Re will lead to an increase in the value of $\beta$, with Condition (5.12) holding. It is clear that there will be no crisis in this model of a turbulent jet.

Computations for the case of $f \equiv 0$ and small negative values of $\Gamma_2'\,(1)$, show that the quantity $\Gamma_1'\,(x_k)$ in (5.10) remains negative within the range $1 < \beta < 1.08$ $(\mathrm{Re}^* \leqslant \mathrm{Re} < 8.2)$. The value of $\Gamma_1'\,(x_k)$ becomes zero when $\beta = 1.08$; $\mathrm{Re} = 8.2$, and in this case $\Gamma_1' \equiv 0$ over the whole zone 1. Therefore the equation for the circulation with boundary conditions

$\Gamma(1) = 0$, $\Gamma'(0) = 0$ has a non-trivial solution. For these parameters we have a bifurcation of the new stationary solution with non-zero azimuthal velocity and zero rotational friction in the plane, i.e. the bifurcation of a weak autorotation.

We note that although we have $\Gamma_1'(0) > 0$ at large $\beta$ and the relation $\Gamma(x)$ has a maximum at $x = x_k$, nevertheless $\Gamma_1(0) > 0$, i.e. we cannot obtain, within the framework of the model of piecewise-constant viscosity, strong autorotation for a purely ascending flow. We did not, however, succeed in proving or disproving this property for an arbitrary function $\varepsilon(x)$.

As we said before, the case of strong autorotation is characterized by the constancy of circulation over the whole zone 1. This makes it possible to determine, using (5.10) and (5.7), the parameter $\beta$ corresponding to the condition of weak autorotation

$$\beta = 2x_k\Gamma_2/[(1 - x_k^2)\Gamma_2' + 2x_k\Gamma_2] \tag{5.13}$$

The bifurcation point obtained corresponds to Condition (5.12), which ceases to hold in the case of a swirling jet.

In order to solve the modes with considerable rotation, we shall waive Condition (5.12), regard the bifurcation number Re as an arbitrary parameter, and determine $\beta$ using (5.13). Then we obtain, in place of a point, a bifurcation curve (curve *1* in Fig.1) on which the quantity $y_2'(1)$ increases from $-\infty$ when $\beta = 1$ and $\mathrm{Re} = \mathrm{Re}_*$.

For $\Gamma$ which are not small, we obtain the pattern shown in Fig.1 where $\overline{\mathrm{Re}} = \mathrm{Re} / \mathrm{Re}_*$, $\Gamma_0 = \Gamma_1(0)/\beta$. The area to the left of curve *1* in the plane $\Gamma_0 = 0$ corresponds to non-swirling flows. At the line *1* we have the bifurcation of the swirling modes (i.e. modes with $\Gamma \neq 0$). The mild character of their excitation implies their stability, and the instability of the initial modes /13/. The families of the lines $\beta = \mathrm{const}$ ($\beta = 1, 3, 10, 30$) and $\Gamma_0 = \mathrm{const}$ ($\Gamma_0 = 0.1, 2, 3$) form a surface symmetrical about the plane $\Gamma_0 = 0$. The surface is bounded by the curve *2* ($\beta = 1$), which corresponds to the crisis involving the loss of existence of the laminar swirling jets induced by the vortex sink in the plane.

In real turbulent swirling jets the parameters $\beta$ and Re are connected with each other, but the relation between them is not known. We shall illustrate this by making the simplest assumption that the relation remains the same as in the case without rotation, when it is given by (5.12). The hypothesis in question corresponds to curve *3* in Fig.1 and the results shown in Fig.2. The quantity $\Gamma_0$ increases as Re increases, and tends to the value $\Gamma_0 \approx 3.8$. The region occupied by the turbulent kernel expands, without, however, reaching the wall; as $\mathrm{Re} \to \infty$, we have $x_k \to 0.33$. An unstable turbulent mode without rotation corresponds to the dashed line.

Fig.3 shows the distribution of the quantities $\bar{y} = y/\beta$ and $\bar{\Gamma} = \Gamma/\beta$ over the angle $\theta$ for a number of typical modes. curve *1* corresponds to the laminar ($y_2'(1) = -460.5$), and curve *2* to the turbulent ($y_2'(1) = -460.5$, $\beta = 20$, $\mathrm{Re} = 268$) non-swirling jets. The curves *3* ($\bar{y}$) and *4* ($\bar{\Gamma}$) correspond to the autorotation mode ($\mathrm{Re} = 140$, $\beta = 12$). We see that even considerable turbulence only slightly affects the flow, while swirling, on the other hand, deforms the pattern sharply with the jet becoming wider and weaker. There is no reverse flow near the axis, which characterizes strongly swirling jets /5/, in case of autorotation.

The upper corner in Fig.3 shows the scheme of meridional motion.

*6. Strong autorotation in the mass force field.* We can achieve a strong autorotation with conditions of adhesion on the plane, by bringing in special mass forces, e.g., the buoyancy forces connected with the thermogravitational mechanism. Let us consider the self-similar problem of thermal convection in the half-space $z \geqslant 0$. In the general case we must attach to the equations of motion (2.1) the equation of heat conduction containing the convection terms. In order to simplify the problem, we shall consider the case when the Prandtl number $\mathrm{Pr} = 0$, assuming that the temperature satisfies the Laplace equation and does not depend on the motion of the fluid.

From the definition of an Archimedean forces $\mathbf{F} = (\rho - \rho_\infty)\mathbf{g}$ and the Boussinesq approximation for the density ($\alpha$ is the volume expansion coefficient) $\rho/\rho_\infty = 1 - \alpha(T - T_\infty)$, it follows that selfsimilar motion under the action of the force (2.3) is possible in the case of a thermal quadrupole (see the second of the papers quoted in the previous footnote). When the fluid is homogeneous, we have the temperature field $T = T_\infty + \gamma(3x^2 - 1)R^{-3}$, corresponding to this quadrupole, which has a sign-alternating overheating $T - T_\infty$ with zero thermal flux. The intensity of the thermal quadrupole is characterized by the parameter $\gamma$, or by the Grasshof number $\mathrm{Gr} = \alpha\gamma g/\nu^2$. When $\gamma > 0$, the zone near the axis is heated $(T > T_\infty)$, and the zone near the wall is cooled $(T < T_\infty)$.

Let us have, in the general case, $T = T_\infty + \gamma\vartheta(x)R^{-3}$. Then, remembering that the mass force $\mathbf{F}$ has a unique non-zero component $F_z$ and using (2.3), we obtain $f_z = \mathrm{Gr}\,\vartheta$. The quantities $f_R$ and $f_\theta$ appearing in (2.8) are determined using the formulas $f_R = xf_z$, $f_\theta = -\sqrt{1-x^2}f_z$. Taking all this into account, we obtain

$$f''' = \mathrm{Gr}\,(x\vartheta' + 3\vartheta) \tag{6.1}$$

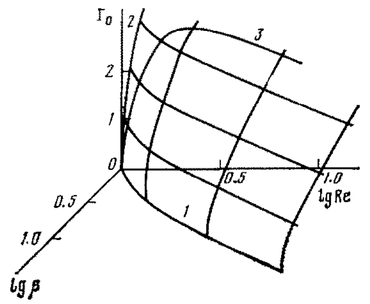


Fig.1

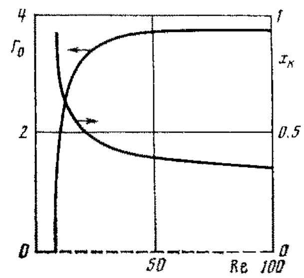


Fig.2

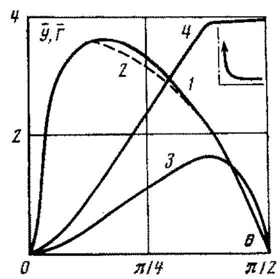


Fig.3

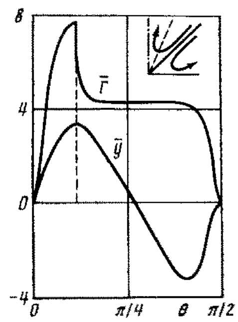


Fig.4

The function $\vartheta(x)$ is given by the equation

$$(1-x^2)\vartheta'' - 2x\vartheta' + 6\vartheta = 0$$

Using its general solution, we obtain the solution of Eq.(6.1)

$$f = \frac{1}{4}\,\mathrm{Gr}\,(1-x^2)^2\left[Mx + N\left(x\ln\frac{1+x}{1-x} - 2\right)\right] \tag{6.2}$$

The conditions of conjugation (5.7)-(5.10) are supplemented by the requirement that the temperature and thermal flux should both be continuous, under the assumption that the ratio of the thermal conductivities is $\lambda_2/\lambda_1 = \beta$. The last assumption obviously cannot be realized physically, but it enables us to solve the temperature problem analytically in full. In a more realistic formulation, which takes into account the empirical data on turbulent heat transfer, we must reject the condition $\mathrm{Pr} = 0$ and solve numerically the complete inter-linked system of equations of motion and energy.

If our aim is to obtain only estimates of the results, then we put, for $x = x_K$,

$$\vartheta_1 = \vartheta_2,\ \vartheta_1' = \beta\vartheta_2 \tag{6.3}$$

The parameters $M$ and $N$ have different values in zones 2 and 1. From the condition of regularity on the axis we have $N_2 = 0$. Writing arbitrarily $M_2 = 1$, we obtain the values of $M_1$ and $N_1$ with the help of (6.3)

$$N_1 = \frac{9}{4}(\beta - 1)\,x_k\,(1 - x_k^2)\left(x_k^2 - \frac{1}{3}\right),$$

$$M_1 = 1 - N_1\left(\ln\frac{1+x_k}{1-x_k} - \frac{2x_k}{x_k^2 - 1/3}\right)$$

after which we construct the solution using the same method as in Sect.5.

If now our aim is only to detect the bifurcation of the autorotation, then we will assume that the quantities $\Gamma$ and $G$ (see (4.5)) are infinitely small, so that we can take $H = f$. The problem will now be reduced to solving Eqs.(5.5) and (5.6) where the quantities $H_i$ are found using (6.2) and (5.2). When $x = 1$, we fix the following quantities: $y_2(1) = \Gamma_2(1) = 0$, $y_2'(1) = -460.5$, $\Gamma_2'(1) = -\delta^2$, $\delta^2 \ll 1$. The parameters $A_1$ and $\beta$ are found by satisfying the conditions of adhesion $y_1(0) = y_1'(0) = 0$, using the method of two-dimensional secants.

The results of computations show that when $Gr < 61.09$, the mode is laminar and $\beta = 1$. For large values of Gr the mode becomes turbulent ($\beta > 1$), but without rotation. When $Gr = 280$, we have detachment and the mode becomes a two-cell mode (see the pattern shown in the upper part of Fig.4). When $Gr = 3920$, $\beta = 4.69$, we have a bifurcation of the strong autorotation. The presence of two cells in meridional flow is obviously the necessary condition for strong autorotation.

Relations $\bar{y}(\theta)$ and $\bar{\Gamma}(\theta)$ corresponding to the bifurcation parameters are shown in Fig.4. The dashed lines denote the boundary of the turbulent kernel ($\theta_K \approx 20°$). The circulation, which varies sharply within the kernel and near it, reaches a constant value. When there is no region with $y < 0$, the quantity $\Gamma$ remains constant. According to (5.3), when the sign of $y$ changes, so does the sign of $\Gamma''$, and a boundary layer appears on the wall as a result, where the value of $\Gamma$ falls to zero. We note that a two-cell mode is also possible when $Pr > 0$.

*7. Discussion.* In connection with the phenomenon of autorotation, the problem arises of the mechanism by which non-zero angular momentum is generated. An analysis of the stationary solutions shows only that when the Reynolds number increases, a stable mode with non-zero rotational velocity branches off the initial mode without rotation, although the moment of external forces remains, as before, equal to zero. In the case of strong autorotation a frictional force moment exists in the plane, as well as an angular momentum flux moving together with the fluid from infinity, i.e. infinity serves as a constant source of momentum.

In the case of weak autorotation, the total angular momentum flux across a hemisphere of any radius is equal to zero. However, the presence of angular momentum implies indirectly that such a flux is present in the transitional process of establishing a new mode, although such a non-stationary process is not discussed here.

Thus, the exchange of angular momentum with infinity is essential for both types of autorotation. Similar phenomena are already known in hydrodynamics. A process, similar to a degree, takes place in a flow past a wing profile. We find, within the model of ideal fluid, that according to the Zhukovskii-Chaplygin mechanism, when a steady state mode is established, the vortices are carried away to infinity, with compensating circulation remaining around the wing. In a viscous fluid we find it necessary, in order for constant circulation to be maintained, that a steady state flux of vorticity to infinity exists. In the first case we have the analogy with weak autorotation, and in the second case with strong autorotation. In the problem of autorotation in a jet /2/, the non-stationary process of establishing a completely defined angular momentum inherent in the secondary mode, was traced explicitly.

Unlike the already known cases, here the necessary condition for the bifurcation of rotational mode is, that the initial irritational flow be turbulent. In this connection we can speak of discovering a new effect, namely the turbulent vortical dynamo.

The fact that strong autorotation really exists is confirmed by the experiment /14/ in which an ascending flow emerging from a point source of heat with external stable stratification (which resembles the quadrupole model discussed here), began to rotate at fairly high Grasshof numbers.

In order to confirm the possibility of weak autorotation, we devised, together with T.V. Li, the following experiment. A motion in a cylindrical tank of 0.6m diameter containing 200 $l$. of water was induced by a radial flow of air in a 5 mm gap between the surface of water and the lid, and the flow was generated by extracting air through a 5 mm diameter hole at the centre of the lid. Observation showed that at low air flow rates the liquid in the tank moves meridionally without rotation. When the air flow rate becomes sufficiently high, the liquid gradually begins to swirl and sometimes it begins to rotate in a direction determinted by the initial small twist. Since the rotational friction between air and water is small, it follows that the conditions of the experiment are close to the problem discussed in Sect.5. These results can be used to support the postulate that a vortical dynamo really exists. It should, however, be stressed that a more-thorough experimental investigation is needed.

From amongst the possible objects to which the results obtained here might be applied, we shall mention the astrophysical jets whose nature remains obscure. From the point of view of the analysis carried out here, we cannot exclude the possibility that the astrophysical jets are generated by an accretion disc contracting towards the centre of gravity, which drags in the surrounding interstellar gas and forms a near-axial jet. The turbulence leads to the formation of a highly viscous kernel and a weak autorotation together with the disc. Thus we find that astrophysical jets are able not only to shed angular momentum from the system, but also to generate a new momentum. Also, it cannot be precluded that the mechanism can also be used to elucidate such a wide prevalence of rotational motion in the cosmos.

The theory of strong autorotation also finds the most likely applications in atmospheric processes. The need for a two-cell structure makes the conditions of appearance of a strong autorotation sufficiently special and hence fairly rare under normal conditions. Nevertheless, in /15/ a special chapter is devoted to describing firestorms where it is shown, in particular, that such a phenomenon occurred in Hiroshima as a result of the atomic bomb. An ascending flow of heated air above a water surface can also generate a whirlwind. A study of the model problem shows that in this case two jets appear, one in air and the other in water. The latter is turbulized earlier, but this does not lead to autorotation. Autorotation occurs when the air jet is also turbulized.

The traditional point of view explains the widespread, and sometimes observed sharp increase in rotational motion in nature (meteorology and in the cosmos), only be constant action of external causes, or by the presence of initial diffuse angular momentum. However, alternative mechanisms of generation of rotation are possible, such as instability, and in particular the turbulent vortical dynamo discussed here.

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Translated by L.K.